

Logic, Reasoning under Uncertainty and Causality

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Abstract

A simple framework for reasoning under uncertainty and intervention is introduced. This is achieved in three steps. First, logic is restated in set-theoretic terms to obtain a framework for reasoning under certainty. Second, this framework is extended to model reasoning under uncertainty. Finally, causal spaces are introduced and shown how they provide enough information to model knowledge containing causal information about the world.

1 Bayesian Probability Theory

It is advantageous to endow plausibilities with an explanatory framework that has a logically intuitive appeal. Such a framework is Bayesian probability theory. Simply put, Bayesian probability theory is a framework that extends logic for reasoning under uncertainty.

1.1 Reasoning under Certainty

Logic is the most important framework of reasoning (under certainty). Here, it is rephrased in set-theoretic terms¹. As will be seen, this facilitates its extension to a framework for reasoning under uncertainty.

Let Ω be a set of **outcomes**, which is assumed to be finite for simplicity. A subset $A \subset \Omega$ is an **event**. Let c , \cup and \cap be the set-operations of **complement**, **union** and **intersection** respectively. Let \mathcal{F} be an **algebra**, i.e. a set of events obeying the axioms

$$\text{A1. } \mathcal{F} \neq \emptyset.$$

$$\text{A2. } A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}.$$

$$\text{A3. } A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}.$$

¹Strictly speaking, this set-theoretic logic is “a logic within logic”, since set theory is based on standard logic.

In this framework, an outcome $\omega \in \Omega$ is a state of affairs and an event $A \in \mathcal{F}$ is a proposition. Hence, a singleton $\{\omega\} \in \mathcal{F}$ is an irreducible (i.e. atomic) proposition about the world. The set-operations c , \cup and \cap correspond to the logical connectives of \neg (negation), \vee (disjunction) and \wedge (conjunction) respectively. They allow the construction of complex propositions from simpler ones. An algebra is a system of propositions that is closed under negation and disjunction (and hence is closed under conjunction as well), i.e. it comprises all propositions that the reasoner might entertain.

Remark 1. *A trivial consequence of the axioms is that both the universal event Ω and the impossible event \emptyset are in \mathcal{F} .*

The objective of logic is to allow the reasoner to conclude the veracity of events given information. Let $\mathcal{V} := \{1, 0, ?\}$ be the set of **truth states**, where 1 is **true**, 0 is **false**, and ? is **uncertain** (but known to be either true or false). From these, $\{1, 0\}$ are called **truth values**. The **truth function** is the set function \mathbf{T} over $\mathcal{F} \times \mathcal{F}$ defined as

$$A, B \in \mathcal{F}, \quad \mathbf{T}(A|B) = \begin{cases} 1 & \text{if } B \subset A, \\ 0 & \text{if } A \cap B = \emptyset, \\ ? & \text{else.} \end{cases}$$

Furthermore, define the shorthand $\mathbf{T}(A) := \mathbf{T}(A|\Omega)$. The quantity $\mathbf{T}(A|B)$ stands for the “truth value of event A given that event B is true”. Accordingly, the knowledge of the reasoner about the facts of the world is represented by his truth function and his algebra. From his point of view, a proposition can be either true, false or uncertain (i.e. having an unresolved truth value given his knowledge). Understanding the definition of the truth function is straightforward. Claiming that an event $B \in \mathcal{F}$ *is true* means that one of its members $\omega \in B$ is the current outcome/state of affairs. Hence the veracity of A given B is evaluated as follows (Figure 1): if A contains every outcome in B then it must be true as well; if A is known not to contain any of B ’s outcome then it must be false; and if A contains only part of B then it cannot be resolved, since knowing that $\omega \in B$ does neither imply that $\omega \in A$ nor $\omega \in A^c$. The definition of a truth space follows.

Definition 1 (Truth Space). A **truth space** is a tuple $(\Omega, \mathcal{F}, \mathbf{T})$ where: Ω is a set of outcomes, \mathcal{F} is an algebra over Ω and $\mathbf{T} : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{V}$ is a truth function.

The intuitive meaning of a truth space is as follows. Nature arbitrarily selects an outcome $\omega \in \Omega$. (This choice is *not* governed by a generative law.) *Subsequently*, the reasoner performs a measurement: he chooses a set B and nature reveals to him whether $\omega \in B$ or not. Accordingly, the reasoner infers the veracity of any event $A \in \mathcal{F}$ by evaluating either $\mathbf{T}(A|B)$ (if $\omega \in B$) or $\mathbf{T}(A|B^c)$ (if $\omega \notin B$).

Several measurements are combined as a conjunction. Thus, if the reasoner learns that ω is in B_1, B_2, \dots , and B_t after performing t measurements, then the truth value is $\mathbf{T}(A|B_1 \cap \dots \cap B_t)$ for any $A \in \mathcal{F}$.

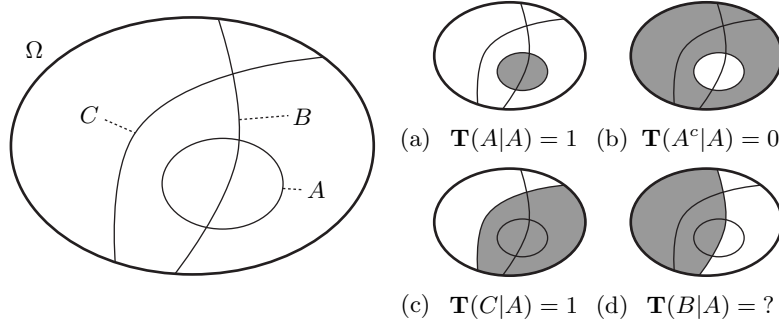


Figure 1: A truth space. It is known that the true outcome $\omega \in \Omega$ is in A . Hence, (a) the event A is true and (b) its complement A^c is false. (c) Any event that contains a true event is true as well. (d) An event that contains only part of a true event is uncertain.

Remark 2. Knowing that $\omega \in \Omega$ does not resolve uncertainty, i.e. $\mathbf{T}(A|\Omega) = ?$ for any $A \in \mathcal{F} \setminus \{\Omega, \emptyset\}$, while knowing that $\omega \in \{\omega\}$ resolves all uncertainty, i.e. $\mathbf{T}(A|\{\omega\}) \in \{0, 1\}$ for any $A \in \mathcal{F}$.

Remark 3. The set relation $B \subset A$ corresponds to the logical relation $B \Rightarrow A$. Since an algebra is an encoding of how sets are contained within each other, it should be clear that an algebra is essentially a system of implications.

1.2 Reasoning under Uncertainty

Unlike logic, Bayesian probability theory allows reasoning under uncertainty. For this end, it provides a consistent mechanism to replace the uncertainty state $?$ with a numerical value in the interval $[0, 1]$ representing degrees of truth, belief or plausibility.

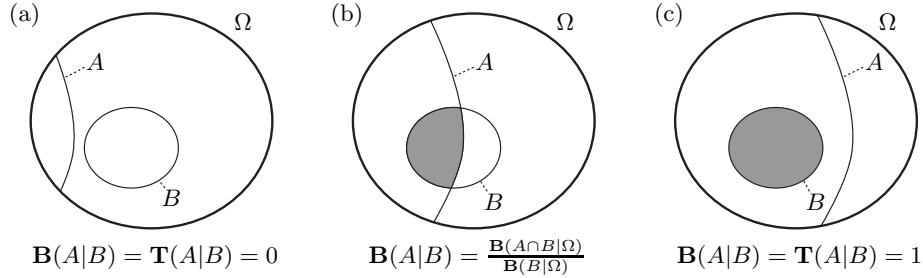


Figure 2: Extension of Truth Function.

The goal is to find a suitable definition of a quantity $\mathbf{B}(A|B)$ meaning “the

degree of belief in event A given that event B is true” that is consistent with the truth function when it is certain, i.e. $\mathbf{B}(A|B) := \mathbf{T}(A|B)$ if $\mathbf{T}(A|B) \in \{0, 1\}$. Consider the three situations in Figure 2. (a) In the case $A \cap B = \emptyset$, we impose $\mathbf{B}(A|B) := \mathbf{T}(A|B) = 0$. (b) In the case $B \subset A$, we impose $\mathbf{B}(A|B) := \mathbf{T}(A|B) = 1$. (c) In the intermediate case where $\mathbf{T}(A|B) = ?$, the event A only partially covers the members of B . If one interprets the quantity $\mathbf{B}(C|D)$ as “the fraction of D contained in C ”, then one can characterize $\mathbf{B}(A|B)$ with the relation

$$\mathbf{B}(A|B) = \frac{\mathbf{B}(A \cap B|\Omega)}{\mathbf{B}(B|\Omega)}$$

as long as $\mathbf{B}(B|\Omega) > 0$. It is easy to see that this formula generalizes correctly to the border cases, since $\mathbf{B}(A|B) = \frac{0}{\mathbf{B}(B|\Omega)} = 0$ when $A \cap B = \emptyset$ and $\mathbf{B}(A|B) = \frac{\mathbf{B}(B|\Omega)}{\mathbf{B}(B|\Omega)} = 1$ when $B \subset A$. Noting that $B = B \cap \Omega$ and rearranging terms, one gets

$$\mathbf{B}(A \cap B|\Omega) = \mathbf{B}(B|\Omega) \mathbf{B}(A|B \cap \Omega).$$

This relation should hold under any restriction to a “universal” set $C \in \mathcal{F}$, not only when it is restricted to Ω . Thus, replacing Ω by C one obtains

$$\mathbf{B}(A \cap B|C) = \mathbf{B}(B|C) \mathbf{B}(A|B \cap C),$$

which is known as the **product rule** for beliefs. Following a similar reasoning, we impose that for any event $A \in \mathcal{F}$, the sum of the degree of belief in A and its complement A^c must be true under any condition B , i.e.

$$\mathbf{B}(A|B) + \mathbf{B}(A^c|B) = 1,$$

which is known as the **sum rule** for beliefs. In summary, we impose the following axioms for beliefs.

Definition 2 (Belief axioms). Let Ω be a set of outcomes and let \mathcal{F} be an algebra over Ω . A set function \mathbf{P} over $\mathcal{F} \times \mathcal{F}$ is a **belief function** iff

- B1. $A, B \in \mathcal{F}$, $\mathbf{B}(A|B) \in [0, 1]$.
- B2. $A, B \in \mathcal{F}$, $\mathbf{B}(A|B) = 1$ if $B \subset A$.
- B3. $A, B \in \mathcal{F}$, $\mathbf{B}(A|B) = 0$ if $A \cap B = \emptyset$.
- B4. $A, B \in \mathcal{F}$, $\mathbf{B}(A|B) + \mathbf{B}(A^c|B) = 1$.
- B5. $A, B, C \in \mathcal{F}$, $\mathbf{B}(A \cap B|C) = \mathbf{B}(A|C) \mathbf{B}(B|A \cap C)$.

Furthermore, define the shorthand $\mathbf{B}(A) := \mathbf{B}(A|\Omega)$. Axiom B1 states that degrees of belief are real values in the unit interval $[0, 1]$. Axioms B2 and B3 equate the belief and the truth function under certainty. Axioms B4 and B5 are the structural requirements under uncertainty discussed above. Accordingly, one defines a belief space as follows.

Definition 3 (Belief Space). A **belief space** is a tuple $(\Omega, \mathcal{F}, \mathbf{B})$ where: Ω is a set of outcomes, \mathcal{F} is an algebra over Ω and $\mathbf{B} : \mathcal{F} \times \mathcal{F} \rightarrow [0, 1]$ is a belief function.

The intuitive meaning of a belief space is analogous to a truth space. Nature arbitrarily selects an outcome $\omega \in \Omega$. *Subsequently*, the reasoner performs a measurement: he chooses a set B and nature reveals to him whether $\omega \in B$ or not. Accordingly, the reasoner infers the degree of belief in any event $A \in \mathcal{F}$ by evaluating either $\mathbf{B}(A|B)$ (if $\omega \in B$) or $\mathbf{B}(A|B^c)$ (if $\omega \notin B$).

Remark 4. The word “subsequently”, that has been emphasized for the second time now, is crucial. When the reasoner performs his measurements, the outcome is already determined.

An easy but fundamental result is that the axioms of belief are equivalent to the axioms of probability². This simple observation is what constitutes the foundation of Bayesian probability theory.

1.3 Bayes’ Rule

We now return to the central topic of this chapter. Suppose the reasoner has uncertainty over a set of competing hypotheses about the world. Subsequently, he makes an observation. He can use this observation to update his beliefs about the hypotheses. The following theorem explains how to carry out this update.

Theorem 1 (Bayes’ Rule). Let $(\Omega, \mathcal{F}, \mathbf{B})$ be a belief space. Let $\{H_1, \dots, H_N\}$ be a partition of Ω , and let $D \in \mathcal{F}$ be an event such that $\mathbf{B}(D) > 0$. Then, for all $n \in \{1, \dots, N\}$,

$$\mathbf{B}(H_n|D) = \frac{\mathbf{B}(D|H_n)\mathbf{B}(H_n)}{\mathbf{B}(D)} = \frac{\mathbf{B}(D|H_n)\mathbf{B}(H_n)}{\sum_m \mathbf{B}(D|H_m)\mathbf{B}(H_m)}.$$

The interpretation is as follows. The H_1, \dots, H_N represent N mutually exclusive **hypotheses**, and the event D represents an new observation or **data**. Initially, the reasoner holds a **prior belief** $\mathbf{B}(H_n)$ over each hypothesis H_n . Subsequently, he incorporates the observation of the event D and arrives at a **posterior belief** $\mathbf{B}(H_n|D)$ over each hypothesis H_n . Bayes’ rule states that this update can be seen as combining the prior belief $\mathbf{B}(H_n)$ with the **likelihood** $\mathbf{B}(D|H_n)$ of observation D under hypothesis H_n . The denominator $\sum_m \mathbf{B}(D|H_m)\mathbf{B}(H_m) = \mathbf{B}(D)$ just plays the rôle of a normalizing constant (Figure 3).

Bayes’ rule naturally applies to a sequential setting. Incorporating a new observation D_t after having observed D_1, D_2, \dots, D_{t-1} updates the beliefs as

$$\mathbf{B}(H_n|D_1 \cap \dots \cap D_t) = \frac{\mathbf{B}_t(D_t|H_n)\mathbf{B}_t(H_n)}{\sum_m \mathbf{B}_t(D_t|H_m)\mathbf{B}_t(H_m)},$$

²More precisely, the axioms of beliefs as stated here imply the axioms of probability for finitely additive measures over finite algebras. Furthermore, the axioms of beliefs also specify a unique version of the conditional probability measure.

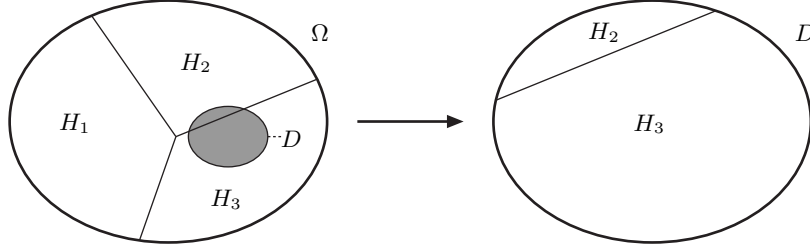


Figure 3: Schematic Representation of Bayes' Rule. The prior belief in hypotheses H_1 , H_2 and H_3 is roughly uniform. After conditioning on the observation D , the belief in hypothesis H_3 increases significantly.

where for the t -th update,

$$\mathbf{B}_t(H_n) := \mathbf{B}(H_n|D_1 \cap \dots \cap D_{t-1}) \quad \text{and} \quad \mathbf{B}_t(D_t|H_n) := \mathbf{B}(D_t|H_n \cap D_1 \cap \dots \cap D_{t-1})$$

play the rôle of the prior belief and the likelihood respectively. Note that

$$\mathbf{B}(D_1 \cap \dots \cap D_t|H_n) = \prod_{\tau=1}^t \mathbf{B}(D_\tau|H_n \cap D_1 \cap \dots \cap D_{\tau-1}),$$

and hence each hypothesis H_n naturally determines a probability measure $\mathbf{B}(\cdot|H_n)$ over sequences of observations.

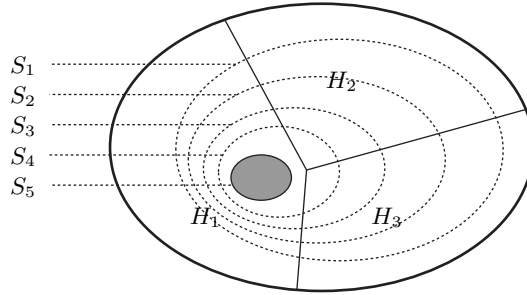


Figure 4: Progressive refinement of the accuracy of the joint observation. The sequence of observations D_1, \dots, D_5 leads to refinements S_1, S_2, \dots, S_5 , where $S_t = D_1 \cap \dots \cap D_t$. Note that $S_5 \subset H_1$ and therefore $\mathbf{B}(H_1|S_5) = 1$, while $\mathbf{B}(H_2|S_5) = \mathbf{B}(H_3|S_5) = 0$.

A smaller event D corresponds to a more “accurate” observation. Hence,

making a new observation D' necessarily improves the accuracy, since

$$D \supset D \cap D'.$$

In some cases, the accuracy of an observation (or sequence of observations) can be so high that it uniquely identifies a hypothesis (Figure 4).

The way Bayes' rule operates can be illustrated as follows. Consider a partition $\{X_1, \dots, X_K\}$ of Ω and let $H_* \in \{H_1, \dots, H_N\}$ be the true hypothesis, i.e. the outcome $\omega \in \Omega$ is drawn obeying propensities described by $\mathbf{B}(\cdot|H_*)$. The X_k represent different observations the reasoner can make. If ω is drawn and reported to be in X_k , then the log-posterior probability of hypothesis H_n is given by

$$\log \mathbf{B}(H_n|X_k) = \underbrace{\log \mathbf{B}(X_k|H_n)}_{l_n} + \underbrace{\log \mathbf{B}(H_n)}_{p_n} - \underbrace{\log \mathbf{B}(X_k)}_c.$$

This decomposition highlights all the relevant terms for understanding Bayesian learning. The term l_n is the log-likelihood of the data X_k . The term p_n is the log-prior of hypothesis H_n , which is a way of representing the relative confidence in hypothesis H_n prior to seeing the data. In practice, it can also be interpreted as (a) a complexity term, (b) the log-posterior resulting from “previous” inference steps, or (c) an initialization term for the inference procedure. The term c is the log-probability of the data, which is constant over the hypotheses, and thus does not affect our analysis. Hence, log-posteriors are compared by their differences in $l_n + p_n$. Ideally, the log-posterior should be maximum for the true hypothesis $H_n = H_*$. However, since ω is chosen randomly, the log-posterior $\log \mathbf{B}(H_n|X_k)$ is a random quantity. If its variance is high enough, then a particular realization of the data can lead to a log-posterior favoring some “wrong” hypotheses over the true hypothesis, i.e. $l_n + p_n > l_* + p_*$ for some $H_n \neq H_*$. In general, this is an unavoidable problem (that necessarily haunts *every* statistical inference method). Further insight can be gained by analyzing the expected log-posterior:

$$\underbrace{\sum_{X_k} \mathbf{B}(X_k|H_*) \log \mathbf{B}(X_k|H_n)}_{L_n} + \underbrace{\log \mathbf{B}(H_n)}_{P_n=p_n} - \underbrace{\sum_{X_k} \mathbf{B}(X_k|H_*) \log \mathbf{B}(X_k)}_C.$$

This reveals³ that, *on average*, the log-likelihood L_n is indeed maximized by $H_n = H_*$. Hence, the posterior belief will, on average, concentrate its mass on the hypotheses having high $L_n + P_n$.

1.4 Conditioning on Events with Zero Belief

There is one technical point that merits closer inspection. Consider two events $A, B \in \mathcal{F}$ such that $B \cap A \neq \emptyset$ but $\mathbf{B}(B) = 0$. One has that

$$\mathbf{T}(A|B) = \begin{cases} 1 & \text{if } B \subset A \\ ? & \text{else} \end{cases} \quad \text{and} \quad \mathbf{B}(A \cap B) = \mathbf{B}(B) \mathbf{B}(A|B) = 0$$

³For p_i, q_i probabilities, $\sum_i p_i \log q_i$ is maximum when $q_i = p_i$ for fixed p_i .

due to the definition of the truth function and due to Axiom B4. From this, we conclude that $\mathbf{B}(A \cap B) = 0$. For $\mathbf{B}(A|B)$ there are two possible cases. If $B \subset A$, then $\mathbf{B}(A|B) = 1$ due to Axiom B2. However, if $B \not\subset A$, then $\mathbf{B}(A|B)$ is independent of the degree of belief $\mathbf{B}(C)$ of any event $C \in \mathcal{F}$. More generally, if $D \in \mathcal{F}$ is such that $\mathbf{B}(B|D) = 0$, then the value of $\mathbf{B}(A|B)$ is independent of the degree of belief $\mathbf{B}(C|D)$ of any event $C \in \mathcal{F}$.

The bottom line is that conditioning on an event with zero belief is a well-defined operation under the belief axioms outlined in Definition 2. This is not so in the case of the probability axioms of measure theory. In measure theory, the probability measure is a global measure μ over \mathcal{F} , i.e. a function assigning probability mass $\mu(A)$ to any event $A \in \mathcal{F}$. However, implicit in this definition is the fact that these masses are measured w.r.t. the certain event Ω . Because of this, the information contained in the probability measure μ is insufficient to uniquely determine the conditional probability measure $\mu(\cdot|B)$ arising from conditioning on an event $B \in \mathcal{F}$ having $\mu(B) = 0$. In contrast, the belief function \mathbf{B} is a well-defined measure w.r.t. any conditioning event $B \in \mathcal{F}$, i.e. assigning probability mass $\mathbf{B}(A|B)$ to any event $A \in \mathcal{F}$.

2 Causality

Suppose there is an unknown cause influencing a result we are waiting for. As soon as we observe the result, we learn something about the unknown cause. However, if instead we decide to interrupt the natural regime of the process by choosing the result ourselves, then our knowledge about the unknown cause will not change. *This is simply because we know that our current actions cannot change the past anymore.* Meanwhile, in both cases, we learn something about the future, i.e. about all the outcomes that will follow the result.

This distinction between belief updates following externally generated observations and internally generated actions is not modeled in Bayesian probability theory. Essentially, the theory lacks the formal tools to deal with indeterminate outcomes chosen by the reasoner himself. This requires introducing additional information to clearly identify the past and the future of choices, or more abstractly speaking, introducing a causal order of events.

3 Causal Spaces

The aim of this section is to introduce causal spaces. Causal spaces contain enough information to characterize the causal structure of a random process.

Let Ω be a finite set of **outcomes**. An **atom set** \mathcal{A} is a partition of Ω , and an **atom** is a member $A \in \mathcal{A}$. Given a set \mathcal{E} of subsets of Ω , define the **algebra generated by** \mathcal{E} , written $\sigma(\mathcal{E})$, as the smallest algebra over Ω containing every member of \mathcal{E} . Furthermore, define the **atom set generated by** an algebra \mathcal{F} , written $\alpha(\mathcal{F})$, as the largest set of atoms containing members of \mathcal{F} . For any set \mathcal{E} of subsets of Ω , we also abbreviate $\alpha(\mathcal{E}) := \alpha(\sigma(\mathcal{E}))$.

Remark 5. In the finite case, it is easily seen that both generated algebras and generated atom sets are unique.

Definition 4 (Primitive Events). Let $E = (E_0, E_1, E_2, \dots, E_N)$ be a finite sequence of subsets of Ω called **primitive events**, where $E_0 := \Omega$, and where for all $n \geq 1$,

$$E_n \notin \sigma\left(\{E_0, E_1, \dots, E_{n-1}\}\right).$$

Furthermore, define $\mathcal{E}_n := \{E_n, E_n^c\}$ and $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_N$ as the sequence of atom sets

$$\mathcal{A}_n := \alpha\left(\{E_0, E_1, \dots, E_n\}\right).$$

This setup is illustrated in Figure 5. The sequence of primitive events is an abstract characterization of a random process that occurs in discrete steps $n = 1, 2, \dots, N$. Each step n is associated with a primitive event E_n representing a basic proposition whose truth value is resolved during this step (and not before!), i.e. step n determines whether the outcome $\omega \in \Omega$ is either in E_n or in E_n^c . The n -th atom set \mathcal{A}_n contains one proposition for each possible path the random process can take. Therefore, after n steps, the process will find itself in one (and only one) of the members in \mathcal{A}_n .

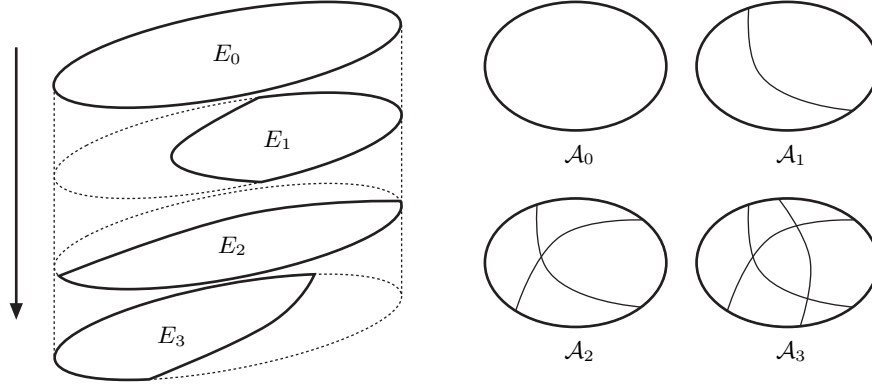


Figure 5: Primitive Events and their Atom Sets.

Remark 6. The condition that E_n cannot be in the algebra generated by the previous events E_0, \dots, E_{n-1} guarantees that E_n adds a new proposition that cannot be expressed in terms of the previous propositions.

The sequence of primitive events $E = (E_1, \dots, E_N)$ can equivalently be represented by any sequence $E' = (E'_1, \dots, E'_N)$ where $E'_n \in \mathcal{E}_n$. Due to this, we will call any member of \mathcal{E}_n primitive event. We introduce causal functions.

Definition 5 (Causal Axioms). Let Ω be a set of outcomes, and let $E = (E_1, \dots, E_N)$ be a sequence of primitive events. A set function \mathbf{C}_n is a **n -th causal function** iff

- C1. $A \in \mathcal{E}_n, B \in \mathcal{A}_{n-1}, \quad \mathbf{C}_n(A|B) \in [0, 1].$
- C2. $A \in \mathcal{E}_n, B \in \mathcal{A}_{n-1}, \quad \mathbf{C}_n(A|B) = 1 \quad \text{if } B \subset A.$
- C3. $A \in \mathcal{E}_n, B \in \mathcal{A}_{n-1}, \quad \mathbf{C}_n(A|B) = 0 \quad \text{if } A \cap B = \emptyset.$
- C4. $A \in \mathcal{E}_n, B \in \mathcal{A}_{n-1}, \quad \mathbf{C}_n(A|B) + \mathbf{C}_n(A^c|B) = 1.$

Hence, \mathbf{C}_n maps $\mathcal{E}_n \times \mathcal{A}_{n-1}$ into $[0, 1]$. A **causal function** over E is a function

$$\mathbf{C}(A|B) = \mathbf{C}_n(A|B), \quad \text{if } A \in \mathcal{E}_n, B \in \mathcal{A}_{n-1},$$

where \mathbf{C}_n is an n -th causal function. Hence, \mathbf{C} maps $\bigcup_n (\mathcal{E}_n \times \mathcal{A}_{n-1})$ into $[0, 1]$.

The intuition behind this definition is as follows. The causal function specifies the knowledge the reasoner has about the evolution of a random process. It specifies the likelihood of a primitive event $A \in \mathcal{E}_n$ to happen after the random process is known to have taken a path $B \in \mathcal{A}_{n-1}$.

By comparing Axioms C1–C4 with Axioms B1–B5 (Section 1.2) of belief functions, we observe the following. First, in contrast to \mathbf{B} , only a subset of combinations $(A, B) \in \mathcal{F} \times \mathcal{F}$ is specified for \mathbf{C} , namely, the ones that chain a history of primitive events $B \in \mathcal{A}_{n-1} \subset \mathcal{F}$ together with the primitive event $A \in \mathcal{E}_n \subset \mathcal{F}$ that immediately follows. Second, Axioms C1–C4 play the same rôle as Axioms B1–B4, namely: (C1) probabilities lie in the unit interval $[0, 1]$; (C2 & C3) probabilities are consistent with the truth function; and (C4) probabilities of complementary events add up to one. No axiom analogous to Axiom B5 is needed for \mathbf{C} .

Putting everything together, one gets a causal space. A causal space contains enough information to derive an associated belief space.

Definition 6 (Causal Space). A **causal space** is a tuple (Ω, E, \mathbf{C}) , where: Ω is a set of outcomes, E is sequence of primitive events, and \mathbf{C} is a causal function over E .

Definition 7 (Induced Belief Space). Given a causal space (Ω, E, \mathbf{C}) , the **induced belief space** is the belief space $(\Omega, \mathcal{F}, \mathbf{B})$ where the algebra \mathcal{F} and the belief function \mathbf{B} are defined as

- i. $\mathcal{F} = \sigma(\{E_0, E_1, \dots, E_N\});$
- ii. $\mathbf{B}(A|B) = \mathbf{C}(A|B), \quad \text{for all } (A, B) \in \bigcup_n (\mathcal{E}_n \times \mathcal{A}_{n-1}).$

Thus, the induced belief space is constructed by generating the algebra \mathcal{F} from the primitive events E , and by equating the belief function \mathbf{B} to the causal function \mathbf{C} over the subset of $\mathcal{F} \times \mathcal{F}$ where \mathbf{C} is defined. The following theorem tells us that this subset is enough to completely determine the whole belief function.

Theorem 2. *The induced belief space exists and is unique.*

Proof. Let $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_N$ denote the sequence of algebras generated as

$$\mathcal{F}_n := \sigma(\{E_0, E_1, \dots, E_n\}).$$

Let $r, s \in \mathbb{N}$, $r \leq s$, be the smallest numbers such that B is \mathcal{F}_r -measurable and A is \mathcal{F}_s -measurable. Let $\mathcal{B} \subset \mathcal{A}_r$ and $\mathcal{A} \subset \mathcal{A}_s$ be the partitions of B and A respectively. Then, $\mathbf{B}(A|B) = 0$ if $A \cap B$ by the belief axioms, and

$$\mathbf{B}(A|B) = \sum_{a \in \mathcal{A}} \mathbf{B}(a|B)$$

otherwise, because the members a of \mathcal{A} are disjoint. For every $a \in \mathcal{A}$, let $b \in \mathcal{B}$ be the unique member of the partition of B such that $a \subset b$. Obviously,

$$\mathbf{B}(a|B) = \mathbf{B}(a|b),$$

because $a \cap B = a \cap b$. Let a^1, a^2, \dots, a^s the unique sequence $a^j \in \mathcal{E}_j$ such that

$$a = a^1 \cap a^2 \cap \dots \cap a^s = \bigcap_{j=1}^s a^j = b \cap \bigcap_{j=r+1}^s a^j,$$

where the last equality comes from $b = a^1 \cap \dots \cap a^r$. Hence,

$$\mathbf{B}(a|b) = \mathbf{B}\left(\bigcap_{j=r+1}^s a^j \middle| b\right) = \prod_{j=r+1}^s \mathbf{B}\left(a^j \middle| b \cap \bigcap_{i=r+1}^{j-1} a^i\right) = \prod_{j=r+1}^s \mathbf{C}\left(a^j \middle| b \cap \bigcap_{i=r+1}^{j-1} a^i\right).$$

The last replacement can be done because $a^j \in \mathcal{E}_j$ and $b \cap \bigcap_{i=r+1}^{j-1} a^i \in \mathcal{A}_{j-1}$. Thus, we have proven the following. First, \mathcal{F} is unique because generated algebras are unique. Second, we have shown, for arbitrarily chosen events $A, B \in \mathcal{F}$, how to reexpress $\mathbf{B}(A|B)$ into an expression involving only terms of the form $\mathbf{C}(C|D)$. Hence, it cannot be that \mathbf{B}, \mathbf{B}' are both consistent with \mathbf{C} and there is $A, B \in \mathcal{F}$ such that $\mathbf{B}(A|B) \neq \mathbf{B}'(A|B)$. \square

We now define the operation that specifies how the knowledge about the random process transforms when the reasoner himself intervenes it.

Definition 8 (Intervention). Given a causal space (Ω, E, \mathbf{C}) and a primitive event $A \in \mathcal{E}_n$ for some $n \in \{1, \dots, N\}$, the **A-intervention** is the causal space (Ω, E, \mathbf{C}') where for all $(B, C) \in \bigcup_n (\mathcal{E}_n \times \mathcal{A}_{n-1})$,

$$\mathbf{C}'(B|C) = \begin{cases} 1 & \text{if } A = B \text{ and } (B \cap C) \notin \{\emptyset, C\}, \\ 0 & \text{if } A = B^c \text{ and } (B \cap C) \notin \{\emptyset, C\}, \\ \mathbf{C}(B|C) & \text{else.} \end{cases}$$

This is an important definition. The reasoner ask himself the question: “How do my beliefs about the world change if I were to choose the truth value of a primitive event?” This is answered by *directly changing the causal function accordingly* (Figure 6). However, this change cannot contradict the logical constraints given by the underlying truth function.

Remark 7. Note that $(B \cap C) \notin \{\emptyset, C\} \Leftrightarrow \mathbf{T}(B|C) = ?$. Hence, an intervention can only affect primitive propositions $B \in \mathcal{E}_n$ that have an unresolved truth value given the history $C \in \mathcal{A}_{n-1}$. Moreover, the intervention resolves the truth value of B . This makes intuitively sense.

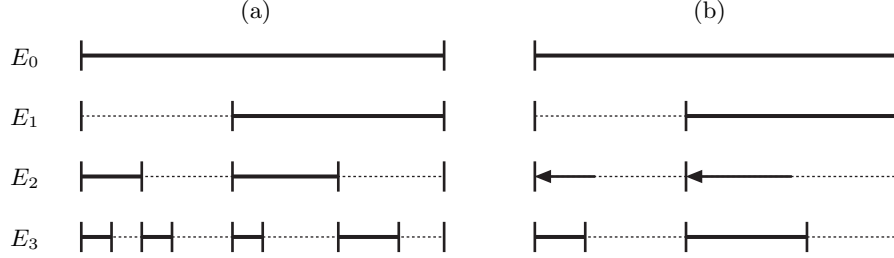


Figure 6: An Intervention. The primitive events $E = (E_0, E_1, E_2, E_3)$ are sets on the unit interval. Panels (a) and (b) show a the causal space before and after an E_2^c -intervention respectively. This representation shows the atom sets \mathcal{A}_0 to \mathcal{A}_3 and conditional probabilities (given by the relative lengths).

We will use the abbreviation \hat{A} to denote A -interventions on a causal space. When the underlying causal space (Ω, E, \mathbf{C}) inducing a belief space $(\Omega, \mathcal{F}, \mathbf{B})$ is clear from the context, then the expression $\mathbf{B}(B|\hat{A})$ denotes the belief $\mathbf{B}'(B|A)$ measured w.r.t. the belief space $(\Omega, \mathcal{F}, \mathbf{B}')$ induced by the A -intervention of (Ω, E, \mathbf{C}) . Furthermore, when $A \in \mathcal{F}$ is an event such that

$$A = \bigcap_{i=1}^I A_i,$$

where each A_i is a primitive event, then the A -intervention is the causal space resulting as the succession of A_i -interventions.

4 Concluding Remarks

We have shown how to derive a simple framework for reasoning under uncertainty and intervention. This is achieved in three steps. First, we have restated logic in set-theoretic terms to obtain a framework for reasoning under certainty.

Second, we have extended this framework to model reasoning under uncertainty. Finally, we have introduced causal spaces and shown how it provides enough information to model knowledge containing causal information about the world.

This framework can be extended in many ways. Importantly, it has been designed to be consistent with the literature on Bayesian statistics [Cox, 1961, Jaynes and Bretthorst, 2003] and the literature on causality based on graphs [Pearl, 2000, Spirtes et al., 2000, Dawid, 2010] and probability trees [Shafer, 1996].

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